SOLUTIONS OF LEIBENSON'S EQUATION WITH PERIODIC BOUNDARY CONDITIONS

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UDC 532.546

1. INTRODUCTION

Solutions of Leibenson's quasi-linear parabolic equation, describing gas filtration in a porous spherical particle during periodic variation of the external gas pressure at its surface, have been investigated numerically in [1]. The formulation of such a problem is associated with modeling of transfer processes in grains of catalyst in nonsteady processes of chemical engineering, in which the reactions can often be made more efficient by increasing the mass-transfer coefficients. The following system of equations has been analyzed:

$$\frac{\partial p}{\partial t} = (b+1)^{-1}\left[\frac{2}{r}\frac{\partial p^{b+1}}{\partial r} + \frac{\partial^2 p^{b+1}}{\partial r^2}\right], \quad b \geqslant 1; \tag{1.1}$$

$$p(R, t) = 1 + A\cos(\omega t + \Delta); \tag{1.2}$$

$$\frac{\partial}{\partial r} p(0, t) = 0; \tag{1.3}$$

$$p(r, 0) = 1, \quad 0 \leqslant r \leqslant R. \tag{1.4}$$

Here p is the gas pressure in the pores; R is the particle's radius. The problem was to investigate (for $b = 1$) the radial pressure distribution as a function of time and frequency ω, the mass-transfer characteristics, etc. The solution was followed over an interval of several periods $T = 2\pi/\omega$ of variation of the boundary conditions. It was shown that the amplitude of pressure pulsations at the center of the sphere decreases with increasing ω, with the pressure at the center averaged over a period first increasing (from $\omega = 1$ to ~ 7) and then slowly decreasing. With increasing frequency, an ever larger region inside the particle becomes ever less sensitive to the external pulsations, but the average pressure in the center may increase in the process. With a further increase in ω, the maximum of the average pressure shifts from the center toward the outer surface of the particle. The mass-transfer coefficient averaged over the period, defined as the gas flux through the particle's surface, increases with increasing ω.

In the present paper, in contrast to [1], we investigate, by the method of a small parameter, mainly steady-state solutions of the system (1.1)-(1.4). An analysis of them shows that some of the features of the behavior of the solution noted in [1] (the nonmonotonic dependence of the average pressure at the center of the particle on frequency ω, the radial distribution of average pressure, and the increase in the mass-transfer coefficient with increasing frequency, in particular) are associated with the shortness of the time interval over which the solution was followed. We show that in the steady state at high frequencies, a constant average pressure, independent of frequency, is established in virtually the entire volume of the particle, except for a thin transition layer near the surface. Features of the relaxation of the solution toward the steady state are studied.

2. STEADY STATE

The initial conditions become unimportant in this case, and we have to satisfy only the boundary conditions (1.2) and (1.3). We assumed an amplitude $A < 1$ (otherwise, we must take into account specific effects associated with reduction of the pressure to zero), and the initial phase is $\Delta = 0$ (in the steady state, one can change to any other value of Δ by a time shift).

Rostov-on-Don. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 6, pp. 84-87, November-December, 1993. Original article submitted June 8, 1992; revision submitted November 8, 1992.

0021-8944/93/3406-0821$12.50

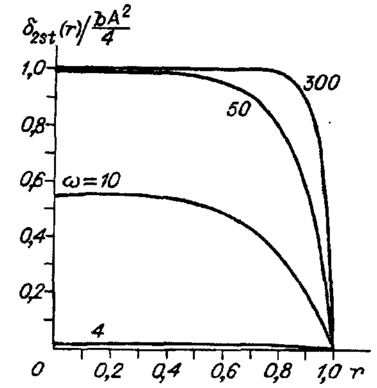

Fig. 1

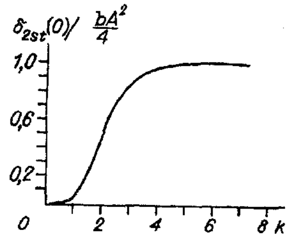

Fig. 2

We seek the solution of the system (1.1)-(1.3) in the series form

$$p_s = 1 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \ldots, \qquad (2.1)$$

where ε is an auxiliary small parameter needed to estimate the order of magnitude of different terms in the equation (we finally set $\varepsilon = 1$ and assume a small amplitude). Substituting (2.1) into (1.1) and equating coefficients of equal powers of ε, as usual, we obtain

$$p_s(r, t) \simeq 1 + \delta_1 + \delta_{2st} + \delta_{2ns} + \ldots, \qquad (2.2)$$

where

$$\delta_1 = \mathrm{Re}\,\{\mu(r)\exp(-i\omega t)\};$$

$$\mu(r) = \frac{AR}{r}\frac{\sin\gamma r}{\sin\gamma R};\quad \gamma = k(i+1);\quad k = (\omega/2)^{1/2};$$

$$\delta_{2st} = \frac{bA^2}{4}\left(1 - \frac{R^2}{r^2}\frac{Z(r)}{Z(R)}\right);$$

$$\delta_{2ns} = \frac{bA^2R^2}{4r}\,\mathrm{Re}\left\{\left[I(r) - \frac{\sin^2\gamma r}{r\sin^2\gamma R} - \left(I(R) - \frac{1}{R}\right)\frac{\sin\lambda r}{\sin\lambda R}\right]\exp(-2i\omega t)\right\};\quad \lambda = \sqrt{2}\,k;$$

$$I(r) = \lambda\int\limits_0^r \frac{\sin^2\gamma s}{\sin^2\gamma R}\,\frac{\sin\lambda(r-s)}{s}\,ds;$$

$$Z(r) = \exp(2kr) + \exp(-2kr) - 2\cos(2kr).$$

We confine ourselves to the quadratic corrections found (for convenience, we divide them into the steady-state component δ_{2st} and the nonsteady component δ_{2ns}), which provide acceptable accuracy for $bA \lesssim 0.5$-0.7. We note that $\delta_{3st} = 0$ and $\delta_{3ns} \sim b^2A^3$ in the next order of perturbation theory.

3. FEATURES OF NONLINEAR CORRECTIONS

Let us consider some properties of the solution obtained. The nonlinearity of the original Eq. (1.1) results in the appearance of an additional constant component of the pressure profile, described by the term δ_{2st}. It is easy to see that the function $\delta_{2st}(r)$ is monotonic and has a plateau at small r, which originates because there are no linear, quadratic, or cubic terms in the series expansion of $Z(r)/r^2$ in powers of r. With increasing k, the plateau becomes ever flatter and its boundary approaches the surface of the sphere, so that as $\omega \to \infty$, the average pressure in virtually the entire volume of the particle (except for the surface layers, where it equals unity) approaches the asymptotic value $1 + (bA^2/4)$. In Fig. 1 we give the function $\delta_{2st}(r)$ for several values of ω (here and in Fig.2 we adopt the normalization $R = 1$).

The averaged pressure at the center is

$$\delta_{2st}(0) = \frac{bA^2}{4}\left(1 - \frac{8k^2R^2}{Z(R)}\right). \tag{3.1}$$

The dependence of $\delta_{2st}(0)$ on k has the typical form of a saturation curve and is given in Fig. 2. These results differ from those given in [1] because at low frequencies the solutions of [1] describe a nearly steady state, while at high frequencies (small T $= 2\pi/\omega$) they describe transitional processes, when the influence of the initial conditions is important (we consider these questions in more detail below). Where the numerical results of [1] and those obtained above are comparable, they are in good agreement. Only at $\omega \sim$ 1-10 do we observe some divergence between our data and those of [1], the reasons for which are unclear. Our numerical calculations confirm the correctness of Eq. (3.1).

For kR $\lesssim$ 1, $\delta_{2st}(0)$ is close to zero and almost no nonlinearity effects appear (at low frequencies the pressure obviously ceases to depend on r).

Let us consider the expression for δ_{2ns}. The integral I(r) is very cumbersome and cannot be expressed in terms of elementary functions, but for large ω it is easy to find its approximate value:

$$I(r) \simeq \frac{\cos^2 \gamma r - \cos \lambda r}{r \sin^2 \gamma R}.$$

The error of this approximation (if r is not too small) is about 15% at $\omega = 30$ and decreases with increasing ω. Note that I(R) $\to -R^{-1}$ as $\omega \to \infty$. The amplitude of oscillations of δ_{2ns} decreases faster than the amplitude of oscillations of δ_1 as the center of the sphere is approached, so that the accuracy of the approximate solution (2.2) depends less on δ_{2ns} than on δ_1 and δ_{2st}.

4. TRANSITIONAL PROCESSES

Let τ_s be the characteristic time of relaxation of the solution toward the steady state (2.2). Let us consider the behavior of the solution at $t < \tau_s$. The result will naturally depend on the initial conditions, which we take, as in [1], in a form such that there are no large-scale averaged motions in the corresponding linear problem. It is easy to follow qualitatively the dynamics of the transitional processes (and the results of [1]) by considering the auxiliary planar problem. For brevity, using the equivalence of filtration and heat-conduction equations, we formulate it in terms of heat transfer. Let the homogeneous medium filling the half-space $0 \le x < \infty$ be heated to the temperature $u(x) = 1$. To describe the temperature distribution in the medium, we must find the steady-state solution of the planar heat-conduction equation

$$\frac{\partial u}{\partial t} = (b+1)^{-1} \frac{\partial^2 u^{b+1}}{\partial x^2} \tag{4.1}$$

under the condition that after a certain time t, which we take as zero, the temperature at the boundary starts to vary as

$$u(0, t) = 1 + A \cos \omega t. \tag{4.2}$$

This is actually the well-known problem of "temperature waves" [2], which differs from the standard formulation by the nonlinearity of Eq. (4.1). Applying the procedure given in Sec. 2 to the system (4.1), (4.2), we obtain

$$\begin{gathered} u \simeq 1 + \varphi_1 + \varphi_{21} + \varphi_{22}, \quad \varphi_1 = A \exp(-kx) \cos(kx - \omega t), \\ \varphi_{21} = \frac{bA^2}{4}\left[1 - \operatorname{erf}\left(\frac{x}{2t^{1/2}}\right) - \exp(-2kx)\right], \\ \varphi_{22} = \frac{bA^2}{2}\left[\exp(-\lambda x)\cos(\lambda x - 2\omega t) - \exp(-2kx)\cos(2kx - 2\omega t)\right]. \end{gathered} \tag{4.3}$$

Comparing (4.3) with (2.2), it is easy to see that the function φ_1 corresponds to δ_1 and can be found from the expression for δ_1 by the limiting transition $R \to \infty$; φ_{22} is obtained similarly from δ_{2ns}. Note that the function φ_{21}, which corresponds formally to δ_{2st}, depends on time; it can be interpreted as the additional "linear" heating of the medium by an amount $bA^2/4$, associated with the fact that the nonlinearity of Eq. (4.1) results in an averaged heat flux from the boundary layer $0 \lesssim x \lesssim (2\omega)^{-1/2}$. Heating of the medium occurs in the entire range $(2\omega)^{-1/2} \lesssim x < \infty$, although temperature oscillations die out near the boundary of the half-space over a distance $d \sim (2/\omega)^{1/2}$. The characteristic time of establishment of oscillations in the region $x \lesssim d$ (which also serves as the heating "source") coincides to order of magnitude with the period $T = 2\pi/\omega$ of variation of the boundary conditions. The characteristic time of "heating" of the medium to a depth l is

$$\tau_h(l) \sim \max(0{,}2l^2;\ T).$$

The nonsteadiness of φ_{21} is related to the infinite size of the system. If we consider the propagation of temperature waves in a layer of large but finite thickness ($L >> d$) (or in a large sphere), then (4.3) has an intermediate asymptotic solution at $T < t << \tau_h(L)$. The time of establishment of steady-state oscillations is $\tau_s(L) \sim \max(T, \tau_h(L)) \sim \tau_h(L)$. If $T < \tau_h$, then at $T < t < \tau_h$ the heat flux associated with φ_{21} decreases with time as $t^{-1/2}$ (at $T \geq \tau_h$ it decreases in a more complicated way). This fact probably accounts for the increase in the average mass-transfer coefficient with increasing frequency in the calculations of [1], in which, we recall, the solution was traced over a time segment inversely proportional to the frequency. The behavior of the function φ_{21} also explains another feature of the solution noted in [1]: the shift of the maximum of the average pressure from the center of the particle toward the surface with increasing ω.

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